

# A nonstandard proof for Szpilrajn's theorem

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## Abstract

Recall that Szpilrajn (1930) ([4], [5]) states that on a given set, any partial order can be extended to a total order on the same set. In this work we give, in the context of the IST theory ([2],[3],[6]), a more constructive proof for this theorem. In addition, we benefit of the tools used to give some other results.

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## 1 Introduction

This paper is placed in the framework of IST ([2],[3],[6]), where to prove the result announced above in the abstract, the main tools are the use, instead of an infinite set  $X$ , of a finite subset  $F = \{x_1, x_2, \dots, x_N\} \subset X$  containing all standard elements of  $X$ , transfer principle and standardization principle. In addition, since we can construct several alternatives of linear order on a given set [1], we give some other applications, which are Propositions 6, 7 and 8. Let us recall the **Standardization principle** : For all formula  $F(Z)$  internal or external, we have:

$$\forall^{st} x \exists^{st} y \forall^{st} z [z \in y \iff z \in x \text{ and } F(z)] \quad (\text{S})$$

where  $x$  is the referential set and  $y$  is a standard set which is the standardized of  $\{z \in x \text{ and } F(z)\}$ . We put  $y = \{z \in x \text{ and } F(z)\}^s$ . Thus, by using the principle (S), we can associate a standard set for any given set. Hence, it is a principle of construction.

**Definition 1** 1) If  $T_1$  and  $T_2$  are subsets of a linearly ordered set, we call  $T_1$  strictly dense (resp. dense) in  $T_2$ , if for every two elements  $a < b$  of  $T_2$  there exists an element  $c \in T_1$  with  $a < c < b$  (resp.  $a \leq c \leq b$ ).

2) Let  $A$  and  $B$  be a given sets.  $A$  is called equivalent to  $B$ , written  $A \sim B$ , if there exists a function  $f : A \longrightarrow B$  which is one-one and onto.

**Notation.** Let  $n$  be a positive integer,  $n \approx +\infty$  denotes that  $n$  is unlimited.

We need to recall that from the main result of [1] we immediately deduce

**Theorem 2** Let  $X$  be any infinite standard set, let  $G = \{\beta_1, \beta_2, \dots, \beta_N\} \subset X$ , ( $N \approx +\infty$ ) be a subset containing all the standard elements of  $X$  and let  $\Omega = \left\{ (\beta_i, \beta_j)_{N \geq i \geq 1, N \geq j \geq i} : \beta_i, \beta_j \in G \right\} \subset X \times X$ . Then, the relation  $\preceq$  defined by  $\Omega^s$  from  $X$  to  $X$  as follows

$$x \preceq y \text{ iff } (x, y) \in \Omega^s$$

is a standard total ordering on  $X$ .

## 2 Main results

**Theorem 3** (Szpilrajn [1930] ) [4] . Let  $(P, \leq)$  be a poset. Then there exists a linear order  $\leq_*$  on  $P$  which contains  $\leq$ , a so-called linear extension of  $\leq$ . In particular: if  $a$  and  $b$  are incomparable elements, this linear order can be chosen in such a way that  $a \leq_* b$  holds.

**Proof.** In order to prove this result we need the following lemmas whose proofs are without use AC [4].

**Lemma 4** [[4], p. 53] . Let  $(P_1, R_1)$  be a poset,  $a$  and  $b$  two incomparable elements of  $P_1$ . Then there exists an order relation  $\widetilde{R}_1$  on  $P_1$  which contains  $R_1$  and in which  $a \widetilde{R}_1 b$  holds.

**Lemma 5** [[4], , p. 27] . Every order  $R_2$  on a finite set  $P_2$  is a subset of a linear order on this set. In other words: Every order relation on a finite set is extendible to a linear order.

Hence, lemma 5 is a partial statement of the theorem of Szpilrajn.

Return to the proof in question. By transfer, we may assume that  $P, \leq, a$  and  $b$  are standard. We distinguish the two following cases.

**A)  $P$  is finite.** By lemma 4, there exists an order relation  $\widetilde{\leq}$  on  $P$  which contains  $\leq$  and in which  $a \widetilde{\leq} b$  holds. Now, by lemma 5 there exists a linear order  $\widetilde{\widetilde{\leq}}$  which

contains  $\widetilde{\leq}$ . By taking  $\widetilde{\leq}$  for  $\leq_*$ , we finish the proof for this case. Indeed, *since*  $\widetilde{\leq}$  is a linear order,  $\leq_*$  is also. Since  $\leq \subset \widetilde{\leq} \subset \widetilde{\leq} = \leq_*$ ,  $\leq_*$  contains  $\leq$ . Moreover, the fact that  $a \widetilde{\leq} b$  entails, since  $\widetilde{\leq} \subset \widetilde{\leq}$ ,  $a \widetilde{\leq} b$  i.e.  $a \leq_* b$ .

**B)** *P is infinite.* Let  $P_1 = \{y_1, y_2, \dots, y_N\}$  be a finite subset of  $P$  containing all standard elements of  $P$ , where  $N$  is necessarily an unlimited integer. In particular,  $a$  and  $b$  belong to  $P_1$ , since they are standard. Now, let us consider the poset  $(P_1, \leq \upharpoonright P_1)$  and denote the order  $\leq \upharpoonright P_1$  by  $R_1$ , where  $\leq \upharpoonright P_1$  is the restriction of  $\leq$  to  $P_1$ .

Then by lemma 4 there exists an order relation  $\widetilde{R}_1$  on  $P_1$  which contains  $R_1$  and in which  $a \widetilde{R}_1 b$  holds. Again, by lemma 5 there exists a linear order  $R$  on  $P_1$  which contains  $\widetilde{R}_1$ . Then  $a R b$ .

Since  $(P_1, R)$  is a chain, we can put  $P_1 = \{x_1, x_2, \dots, x_N\}$  ( $x_i \in \{y_1, y_2, \dots, y_N\}$  for  $1 \leq i \leq N$ ), such that  $x_1 R x_2, x_2 R x_3, x_3 R x_4, \dots, x_{N-1} R x_N$ . We notice here that  $a$  (resp.  $b$ ) corresponds to  $x_{i_1}$  (resp. to  $x_{i_2}$ ), where  $i_1, i_2$  are in  $\{1, 2, \dots, N\}$  with  $x_{i_1} R x_{i_2}$  since  $a \widetilde{R}_1 b$ . Let  $G = \{(x_i, x_j)_{N \geq i \geq 1, N \geq j \geq i} \mid x_i, x_j \in P_1\}$  be the graph of  $(P_1, R)$ , where saying that  $(x_i, x_j) \in G$  is the same to saying that  $x_i R x_j$ . Let  $G^s \subset P \times P$  be the standardization of  $G$ . It is known that  $G^s$  defines a relation  $\Gamma$  from  $P$  to  $P$  as follows

$$x \Gamma y \text{ iff } (x, y) \in G^s.$$

According to theorem 2,  $\Gamma$  is a total order on  $P$ .

Now, we prove the remainder of the theorem. Let  $(x, y) \in P^2$  be a standard element verifying  $x \leq y$ . Then we have successively  $x R_1 y, x \widetilde{R}_1 y, x R y$ . Then  $(x, y) \in G$  and therefore  $(x, y) \in G^s$  because  $(x, y)$  is standard. Hence  $x \Gamma y$ . So

$$\forall^{st} (x, y) \in P^2 [(x \leq y) \implies x \Gamma y].$$

Then by transfer,  $\forall (x, y) \in P^2 [(x \leq y) \implies x \Gamma y]$ . Hence,  $\Gamma$  contains  $\leq$ . From what precedes, we have concerning  $(a, b)$ :  $a R b$ . This shows that  $(a, b) \in G$  i.e.  $(a, b) \in G^s$  since  $(a, b)$  is standard. Consequently,  $a \Gamma b$ . Now, by taking  $\Gamma$  for  $\leq_*$  we finish the proof for this case. By transfer, we conclude for all  $P$ . ■

**Proposition 6** *Let  $X$  be any set. Suppose that  $A$  and  $B$  are nonempty disjoint subsets of  $X$ . Then we can provide  $X$  by a total ordering  $\preceq$  such that*

$$\forall x, y \in X [(x \in A \text{ and } y \in B) \implies x \preceq y].$$

**Proof.** By transfer, we suppose  $X, A$  and  $B$  are standard. Let

$$F = \{\beta_1, \beta_2, \dots, \beta_N\} \subset X$$

be a finite subset containing all standard elements of  $X$ . Suppose that this numbering of elements of  $F$  has been made so that elements of  $A \cap F$  are before those of  $B \cap F$ . Now, by theorem 2 and transfer we finish the proof. ■

**Proposition 7** *Let  $X$  be any set. Let  $S$  be a system of mutually disjoint subsets of  $X$ . Then we can provide  $X$  by a total ordering such that*

$$\forall A, B \in S [A < B \text{ or } B < A].$$

**Proof.** Assume, by transfer, that  $X$  and  $S$  are standard. Let  $F = \{x_1, x_2, \dots, x_{\omega_1}\}$ ,  $\omega_1 \cong +\infty$  (resp.  $S = \{A_1, A_2, \dots, A_{\omega_2}\}$ ,  $\omega_2 \cong +\infty$ ) be a finite subset of  $X$  (resp. of  $S$ ) containing all standard elements of  $X$  (resp. of  $S$ ). For  $j = 1, 2, \dots, \omega_2$ , we put

$$F \cap A_j = \{y_{j,1}, y_{j,2}, \dots, y_{j,n_j}\}$$

which is a finite set containing all standard elements of  $A_j$ . Put  $L = F \setminus \bigcup_{j=1}^{\omega_2} (F \cap A_j) = \{y_1, y_2, \dots, y_s\}$ .

Now, we rearrange elements of  $F$  as follows : We begin by elements of  $F \cap A_1$ , after those of  $F \cap A_2$ , after those of  $F \cap A_3$  and so on. Thus  $F$  becomes

$$F = \{y_{1,1}, y_{1,2}, \dots, y_{1,n_1}, y_{2,1}, y_{2,2}, \dots, y_{2,n_2}, \dots, y_{\omega_2,1}, y_{\omega_2,2}, \dots, y_{\omega_2,n_{\omega_2}}\} \cup \{y_1, y_2, \dots, y_s\}$$

where  $\sum_{j=1}^{\omega_2} n_j + s = \omega_1$ . Now, by theorem 2 and transfer we finish the proof. ■

**Proposition 8** *Let  $X$  and  $Y$  be any sets. If  $X \sim Y$  then we can provide  $X \cup Y$  by a total ordering such that  $X$  becomes strictly dense in  $Y$ .*

**Proof.** By transfer, we may assume that  $X$  and  $Y$  are standard. Since  $X \sim Y$  then there exists a mapping  $\varphi : Y \rightarrow X$  which is one-one and onto. By transfer  $\varphi$  is standard. Let  $F_Y = \{y_1, y_2, \dots, y_N\}$  be a finite subset of  $Y$  containing all standard elements of  $Y$ . Then  $\varphi[F_Y] = \{\varphi(y_1), \varphi(y_2), \dots, \varphi(y_N)\}$  is a finite subset of  $X$  containing all standard elements of  $X$ . Let us put

$$F_{X \cup Y} = \{y_1, \varphi(y_1), y_2, \varphi(y_2), y_3, \varphi(y_3), y_4, \varphi(y_4), \dots, \varphi(y_{N-1}), y_N, \varphi(y_N)\}.$$

Then  $F_{X \cup Y}$  is a finite subset of  $X \cup Y$  containing all the standard elements of  $X \cup Y$ . As well as in the theorem 2, we construct from  $F_{X \cup Y}$  the set  $G$  and thereafter the set  $G^s$  which is a standard linear order in  $X \cup Y$ .

Let  $y_i$  and  $y_j$  be standard elements of  $Y$  such that  $y_i \prec y_j$  in  $G^s$  which is equivalent to  $i < j$ . Then we have  $y_i < \varphi(y_i) < y_j$ , where  $\varphi(y_i)$  is a standard element of  $X$ . Therefore

$$\forall^{st} (\alpha, \beta) \in Y^2 \exists^{st} \gamma \in X [\alpha < \beta \implies \alpha < \gamma < \beta].$$

Which, by transfer, entails  $\forall (\alpha, \beta) \in Y^2 \exists \gamma \in X [\alpha < \beta \implies \alpha < \gamma < \beta]$ . Consequently  $X$  is strictly dense in  $Y$ , where  $X$  and  $Y$  are subsets of  $X \cup Y$  which is linearly ordered by  $G^s$ . ■

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